Proof of Certain Inequalities and Functional Equations

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A B S T R A C T
In this article, a new method is used to solve some inequalities, and it can also be used to solve some geometric problems.

A R T I C L E I N F O
Article history:
Received 25 Dec 2023
Received in revised form 26 Jan 2024
Accepted 28 Feb 2024

Keywords: Inequality, equivalent, multiplicative.

1. $a \cdot b \cdot c = 1$ all that satisfy the condition are greater than zero a, b, c prove this inequality for numbers:

$$\left(a-1+\frac{1}{b}\right)\left(b-1+\frac{1}{c}\right)\left(c-1+\frac{1}{a}\right)\le 1$$

Proof: $b-1+\frac{1}{c}=b\left(1-\frac{1}{b}+\frac{1}{bc}\right)=b\left(1+a-\frac{1}{b}\right)\Rightarrow\left(b-1+\frac{1}{c}\right)\left(a-1+\frac{1}{b}\right)=b\left(a^2-\left(1-\frac{1}{b}\right)^2\right)\le ba^2;$

Likewise $\left(c-1+\frac{1}{a}\right)\left(b-1+\frac{1}{c}\right)\le ac^2$, $\left(a-1+\frac{1}{b}\right)\left(c-1+\frac{1}{a}\right)\le cb^2 \Rightarrow$

(multiply all three) $\Rightarrow \left(a-1+\frac{1}{b}\right)^2\left(b-1+\frac{1}{c}\right)^2\left(c-1+\frac{1}{a}\right)^2\le ba^2\cdot ac^2\cdot cb^2 = 1$

$$\left(a-1+\frac{1}{b}\right)\left(b-1+\frac{1}{c}\right)\left(c-1+\frac{1}{a}\right)\le 1$$

2. $\forall a,b,c>0$ prove the inequality for: $\frac{a}{\sqrt{a^2+8bc}}+\frac{b}{\sqrt{b^2+8ca}}+\frac{c}{\sqrt{c^2+8ab}}\ge 1$ (A Ibragimov, 2023)

Proof: 1)

$$8a^{\frac{2}{3}}bc = 2\cdot b^{\frac{2}{3}}\cdot c^{\frac{2}{3}}\cdot 4a^{\frac{2}{3}}\cdot b^{\frac{1}{3}}\cdot c^{\frac{1}{3}} \le \left(b^{\frac{4}{3}}+c^{\frac{4}{3}}\right)\left(a^{\frac{4}{3}}+a^{\frac{4}{3}}+b^{\frac{4}{3}}+c^{\frac{4}{3}}\right) = \left(a^{\frac{4}{3}}+b^{\frac{4}{3}}+c^{\frac{4}{3}}\right)^2 - \left(a^{\frac{4}{3}}\right)^2 \Rightarrow$$

$$a^{\frac{1}{3}\sqrt{a^2+8bc}} \le a^{\frac{4}{3}}+b^{\frac{4}{3}}+c^{\frac{4}{3}} \Rightarrow$$

$$\begin{cases} \dfrac{a}{\sqrt{a^2+8bc}} \ge \dfrac{a^{\frac{4}{3}}}{a^{\frac{4}{3}}+b^{\frac{4}{3}}+c^{\frac{4}{3}}} \\ \dfrac{b}{\sqrt{b^2+8ba}} \ge \dfrac{b^{\frac{4}{3}}}{a^{\frac{4}{3}}+b^{\frac{4}{3}}+c^{\frac{4}{3}}} \\ \dfrac{c}{\sqrt{c^2+8bc}} \ge \dfrac{c^{\frac{4}{3}}}{a^{\frac{4}{3}}+b^{\frac{4}{3}}+c^{\frac{4}{3}}} \end{cases} \Rightarrow \frac{a}{\sqrt{a^2+8bc}}+\frac{b}{\sqrt{b^2+8ca}}+\frac{c}{\sqrt{c^2+8ab}} \ge 1$$

2) $$\frac{a}{\sqrt{a^2+8bc}}+\frac{b}{\sqrt{b^2+8ca}}+\frac{c}{\sqrt{c^2+8ab}} = \frac{1}{\sqrt{1+\frac{8bc}{a^2}}}+\frac{1}{\sqrt{1+\frac{8ca}{b^2}}}+\frac{1}{\sqrt{1+\frac{8ab}{c^2}}}$$

The given inequality is equivalent to the following theorem:

Theorem: If $t_1\cdot t_2\cdot t_3 = 512$ there is $\dfrac{1}{\sqrt{1+t_1}}+\dfrac{1}{\sqrt{1+t_2}}+\dfrac{1}{\sqrt{1+t_3}} \ge 1$;

Reminder: $\forall a,b,c>0$ and $\lambda \ge 0$ $\dfrac{a}{\sqrt{a^2+\lambda bc}}+\dfrac{b}{\sqrt{b^2+\lambda ca}}+\dfrac{c}{\sqrt{c^2+\lambda ab}} \ge \dfrac{3}{\sqrt{1+\lambda}}$;

3. Optional $x,y,z,t\in R$ and $f(xy-zt)+f(xt+yz)=(f(x)+f(z))(f(y)+f(t))$ $f:R\to R$ find functions that satisfy the condition.[1]

The solution: $x,y,z,t\in R$ for arbitrary numbers $f(xy-zt)+f(xt+yz)=(f(x)+f(z))(f(y)+f(t))$ (1) equality is given.

(1) equality $x=y=z=0$ that $2f(0)=2f(0)\cdot(f(0)+f(t))$ equality, and from that $f(0)=0$ or $f(0)=\dfrac{1}{2}$.

If $f(0)=\dfrac{1}{2}$ and $\Rightarrow$ $f(0)+f(t)=1 \Rightarrow f(t)\equiv\dfrac{1}{2}$;

After $f(0)=0$. In that case $z=t=0$ (1) by putting $f(xy)=f(x)\cdot f(y)$ that is, we form the multiplicative function f, $f(1)=f^2(1)\Rightarrow f(1)=1$ or $f(1)=0$. If $f(1)=0$ and $f(x)=f(x)\cdot f(1)=0\Rightarrow f(x)\equiv 0, x\in R$;

$f(0)=0$ and $f(1)=1$ Now all that remains is to find f that satisfies equation (1).

(1) and $x=0$ $y=t=1$ $f(-z)+f(z)=2f(z)$, $f(-z)=f(z)$ we find that equality is an f-even function.

(1) in Eq $y=t=z=1$ let's say $f(x-1)+f(x+1)=2(f(x)+1)$ will be.

[1] Курбон Останов, Ойбек Улашевич Пулатов, Джумаев Максуд, «Обучение умениям доказать при изучении курса алгебры,» *Достижения науки и образования,* т. 2 (24), № 24, pp. 52-53, 2018.

From this equation $f(0)=0,\ f(1)=1$ by induction, given that $n\in Z$ and we find that $f(n)=n^2$.

Also $x\in Q$ $f(x)=x^2$. (Because f(x) is multiplicative $\Rightarrow$ $x=\frac{p}{g};\Rightarrow f\left(\frac{p}{g}\right)\cdot f(g)=f(p)\Rightarrow f\left(\frac{p}{g}\right)=\frac{f(p)}{f(g)}=\frac{p^2}{g^2}$)

(1) from the multiplicity of the function in Eq $f(x^2)=f^2(x)\geq 0$, $f(y)\geq 0; y\in R$ $t=x, y=z$ $\Rightarrow$ $f(x^2+y^2)=(f(x)+f(y))^2;\Rightarrow f(x^2+y^2)\geq f^2(x)=f(x^2)\Rightarrow u\geq v\geq 0$, $f(u)=f(v)$, that is, the function f is increasing in positive real numbers. $\forall x\in R$ $f(x)=x^2$ we show that.

Suppose for some x $f(x)\neq x^2$ and $f(x)<x^2$ let it be Then some rational number a is found $x>a>\sqrt{f(x)}$. $\Rightarrow f(a)=a^2>f(x)$. But $f(a)=f(x)$, because f(x) is increasing. ($x\geq 0$).

After $f(x)>x^2$ we will show that it cannot be.

$f(x)=x^2, x\in R$ so it is.

4. $n\in N$ and $x_1,x_2,...,x_n\in R$, $x_1\leq x_2\leq ...\leq x_n$

(a) $\left(\sum_{i=1}^{n}\sum_{j=1}^{n}|x_i-x_j|\right)^2\leq\frac{2(n^2-1)}{3}\sum_{i=1}^{n}\sum_{j=1}^{n}(x_i-x_j)^2$ prove the inequality.

(b) For the fulfillment of Eq $x_1,x_2,...,x_n$ prove that it is necessary and sufficient that the numbers form an arithmetic progression.

Proof: We use the following lemma in the proof.

Lemma: If $x_1\leq x_2\leq ...\leq x_n$ and $\sum_{i,j=1}^{n}|i-j||x_i-x_j|=\frac{n}{2}\sum_{i,j=1}^{n}|x_i-x_j|$.

Proof: Without prejudice to generality $i>j$ let's say. $\Rightarrow$

$$\sum_{i,j=1}^{n}|i-j||x_i-x_j|=(n-1)(x_n-x_1)+(n-2)(x_n-x_2)+(n-3)(x_n-x_3)+...+(x_n-x_{n-1})+$$
$$(n-2)(x_{n-1}-x_1)+(n-3)(x_{n-1}-x_2)+...+(x_{n-1}-x_n)+...+(x_2-x_1)=$$
$$x_n((n-1)+(n-2)+...+1)+x_{n-1}((n-2)+...+1-1)+$$
$$x_{n-2}((n-3)+...+1-1-2)+...-x_1(1+2+...+(n-1))=$$
$$x_n\frac{n(n-1)}{2}+x_{n-1}\frac{n(n-3)}{2}+x_{n-2}\frac{n(n-5)}{2}+...-x_1\frac{n(n-1)}{2}=$$
$$\frac{n}{2}((n-1)x_n+(n-3)x_{n-1}+...-(n-3)x_2-(n-1)x_1)=\frac{n}{2}\sum_{i,j=1}^{n}|x_i-x_j|.$$

The lemma is proved. Now we proceed to prove the given inequality.

$\sum_{i=1}^{n}\sum_{j=1}^{n}|x_i-x_j|=2\sum_{1\leq j\prec i\leq n}|x_i-x_j|\Rightarrow\left(\sum_{1\leq j\prec i\leq n}|x_i-x_j|\right)^2\leq\frac{n^2-1}{3}\sum_{1\leq j\prec i\leq n}|x_i-x_j|^2$ $(**)$ it is enough to prove that.

Let's say $\frac{x_i-x_j}{i-j}=a_{ij}$. Without prejudice to the generality $i>j\Rightarrow\left(\sum_{1\leq j\prec i\leq n}|i-j|a_{ij}\right)^2\leq\frac{n^2-1}{3}\sum_{1\leq j\prec i\leq n}(i-j)^2a_{ij}^2$ it is sufficient to prove.

According to the Cauchy-Buniakovsky inequality

$$\left(\sum_{1\le j\prec i\le n}(i-j)^2 a_{ij}^{\ 2}\right)\cdot\left(\sum_{1\le j\prec i\le n}(i-j)^2\right)\ge\left(\sum_{1\le j\prec i\le n}(i-j)^2 a_{ij}\right)^2=\frac{n^2}{4}\left(\sum_{1\le j\prec i\le n}(x_i-x_j)\right)^2 \quad (1)$$

$\sum_{1\le j\prec i\le n}(i-j)^2$ we calculate.

$$\sum_{1\le j\prec i\le n}(i-j)^2=(n-1)^2+2(n-2)^2+3(n-3)^2+...+(n-1)(n-(n-1))^2=$$
$$=n^2(1+2+...+(n-1))-2n(1^2+2^2+...+(n-1)^2)+(1^3+2^3+...+(n-1)^3)=$$
$$=\frac{n^3(n-1)}{2}-2n\frac{(n-1)n(2n-1)}{6}+\frac{n^2(n-1)^2}{4}=\frac{n^2(n^2-1)}{12}$$

$(1)\Rightarrow\left(\sum_{1\le j\prec i\le n}(i-j)^2 a_{ij}^{\ 2}\right)\frac{n^2-1}{3}\ge\left(\sum_{1\le j\prec i\le n}(x_i-x_j)\right)^2 \ \Rightarrow(**)$ proved.

(b) $(1)\Rightarrow\ \frac{(i-j)^2 a_{ij}^{\ 2}}{i-j}=a_{ij}^{\ 2}$ is appropriate when mutually equal $\Rightarrow x_i-x_j=(i-j)a_{ij}=(i-j)d$ d=const $\Rightarrow\ j=1\ \ x_i-x_1=(i-1)d\Rightarrow x_n=x_1+(n-1)d$.

proved.

5. $\frac{a^2}{2ab^2-b^3+1}=k$ to find all N solutions of the equation. $a^2-2ab^2k+kb^3-k=0\ \ (*)$

we solve the quadratic equation in natural numbers. Suppose that one root of this equation is N $\Rightarrow$ $a_1+a_2=2b^2k\in N$.

So, $D=4b^4k-4(kb^3-k)=(2b^2k-b)^2+4k-b^2;$

$\forall k,b\in N$ for $(2b^2k-b-1)^2<D=(2b^2k-b)^2+4k-b^2\le(2b^2k-b+1)$ it is not difficult to show that.

$D=(2b^2k-b+1)^2$ and $b=1$ must be $\Rightarrow\frac{a^2}{2ab^2-b^3+1}=\frac{a^2}{2a}=\frac{a}{2}\Rightarrow$ $(a,b)=(2m,1)$, $m\in N$ $b\ne1$ da $(2b^2k-b+1)^2<D=(2b^2k-b)^2+4k-b^2<(2b^2k-b+1)^2.\quad(**)$

$(*)$ For a quadratic equation to have a N solution, the discriminant must be an integer square. $\Rightarrow$

$\Rightarrow(**)\ \ D=(2b^2k-b)^2\Rightarrow 4k-b^2=0\Rightarrow b^2=4k\Rightarrow b^2=4l^2$

$M=l^2$. So, $b=2l\Rightarrow a_{1,2}=\frac{2b^2k\pm(2b^2k-b)}{2}$

$$a_1=\frac{2b^2k+2b^2k-b}{2}=\frac{4b^2k-b}{2}=\frac{4\cdot4l^2\cdot l^2-2l}{2}=8l^4-l$$

$$a_2=\frac{2b^2k-2b^2k+b}{2}=\frac{b}{2}=\frac{2l}{2}=l$$

$(a,b)=(2m,1),(8l^4-l,2l),(l,2l).$

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