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On A Method for Solving Parametric Equations With Respect to a Parameter

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A B S T R A C T

In this article, the parametric equation becomes more complicated with respect to the main variable (unknown), and if it is a simpler equation with respect to the parameter, then when it is required to solve the equation for x, the variables x and a are considered equal in it, and it is simpler than them, the equation to be solved is chosen. Then we solve it with respect to the parameter a and find its solution depending on x. From the solution found, x is expressed in terms of a and it is shown that it is a solution to the original equation.

A R T I C L E I N F O

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The solution of parametric equations does not have a common understanding in the methodological literature. For example, in [5], if the expression of the main unknown, depending on the parameter, turns the equation into reality for possible values of the parameter, then it is considered a solution. In [6], the set of possible pairs of the main variable and parameter is understood, which turn the given equation into reality. In [4], the notion of a general solution was introduced.

We understand the solution of parametric equations as in [5]. It is known that a parametric equation (basic and dependent on a parameter) can be considered as two equations, that is, an equation for the main unknown or an equation for a parameter. If the equation for the key variable (unknown) is complex, but the equation for the parameter is simpler, then you can solve for the parameter and then find the key of the unknown. For example, such equations were considered in [1], [4], [5]. When solving the equation *F(x,a)=0* (1) with respect to the underlying unknown *x* is complex, we treat it as an equation with respect to the parameter *a* and solve it with respect to the parameter $a = \varphi(x)$ (2) and again solve (2) with respect to *x*, we find solutions of this equation *x=g(a)* (3) and show that (3) satisfies (1). Let's demonstrate this method with several examples.

Example 1: Solve the equation$(a^2-4)\cdot x = a-2$.

Solution: *Method 1:* Let's give this equation the form $x\cdot a^2-4x-a+2=0$. Considering this equation as quadratic with respect to a, $xa^2-a-(4x-2)=0$, we find its roots: by finding x from $x\neq 0, a_{1.2}=$

$$\frac{1\pm\sqrt{1+4x(4x-2)}}{2x}=\frac{1\pm\sqrt{16x^2-8x+1}}{2x}=\frac{1\pm\sqrt{(4x-1)^2}}{2x}=\frac{1\pm|4x-1|}{2x}=|4x-1|=\begin{cases}4x-1, x\geq\frac{1}{4}\\-4x+1, x<\frac{1}{4}\end{cases}\Rightarrow$$

$$\begin{cases}a_1=\frac{1+4x-1}{2x}=2\\a_2=\frac{1-4x+1}{2x}=\frac{1-2x}{x}\end{cases}\Rightarrow\ a=\frac{1-2x}{x}\ (a+2)\cdot x=1\Rightarrow\begin{cases}a=2, 4x-4x-2+2=0\\a=-2,\ (-2+2)\cdot x=1\\a\neq\pm 2,\ (a+2)\cdot x=1\end{cases}\Rightarrow\begin{cases}0\cdot x=0\\0\cdot x=1\\x=\frac{1}{a+2}\end{cases}$$

Answer: for $a=2$ there are infinitely many solutions, for $a=-2$ there are no solutions, for $a\neq\pm 2$ da $=\frac{1}{a+2}$.

Method 2: We solve the equation $(a^2-4)\cdot x = a-2$ by setting conditions for a.

When solving this equation, consider the following cases.

1) $a^2-4=0$, i.e. $a=2$ and $a=-2$.

If $a=-2$, then the equation takes the form $0\cdot x=-4$ and has no solution.

If $a=2$, then we get $0\cdot x=0$ x is an arbitrary number.

2) If $a\neq\pm2$ then $x=\frac{1}{a+2}$.

Answer: If $a=-2$, then there will be no solution. If $a=2$, x - is an arbitrary number.

If $a\neq\pm2$, then $=\frac{1}{a+2}$.

Example 2: Solve the equation. $x^4-2bx^2-x+b^2-b=0$

Solution: Since this equation is a quadratic equation in x, we find its roots by treating $b^2-(2x^2+1)b+x^4-x=0$ as a quadratic equation in b: $b_{1.2}=\frac{2x^2+1\pm\sqrt{(2x^2+1)^2-4(x^4-x)}}{2}=\frac{2x^2+1\pm\sqrt{4x^2+4x+1}}{2}=\frac{2x^2+1\pm(2x+1)}{2}$ i.e. $b_1=x^2+x+1$, we get $b_2=x^2-x$ $(*)$. Now let's solve these equations for x.

1. $b=x^2+x+1, x^2+x+1-b=0$, when this equation looks like $b\geq\frac{3}{4}$ $x_{1.2}=\frac{-1\pm\sqrt{1-4(1-b)}}{2}=\frac{-1\pm\sqrt{4b-3}}{2}$ has solutions.

2. $b=x^2-x$, $x^2-x-b=0, b\geq-\frac{1}{4}, x_{3.4}=\frac{1\pm\sqrt{1+4b}}{2}$ has solutions. $(*)$ we will show that the found solutions of the equations satisfy the given equation. $x_1=\frac{-1+\sqrt{4b-3}}{2}$,

$$\left(\frac{-1+\sqrt{4b-3}}{2}\right)^4-2b\cdot\left(\frac{-1+\sqrt{4b-3}}{2}\right)^2-\frac{-1+\sqrt{4b-3}}{2}+b^2-b=$$

$$=\left(\frac{-1+\sqrt{4b-3}}{2}\right)^2\cdot\left(\frac{-2+4b-2\sqrt{4b-3}}{4}-2b\right)-\frac{2b^2-2b+1+\sqrt{4b-3}}{2}=$$

$$=\frac{2b-1-\sqrt{4b-3}}{2}\cdot\frac{-2b-1-\sqrt{4b-3}}{2}-\frac{2b^2-2b+1-\sqrt{4b-3}}{2}=$$

$$=\frac{1+\sqrt{4b-3}-2b}{2}\cdot\frac{1+\sqrt{4b-3}+2b}{2}-\frac{2b^2-2b+1-\sqrt{4b-3}}{2}=$$

$$=\frac{2b^2-2b+1-\sqrt{4b-3}}{2}-\frac{2b^2-2b+1-\sqrt{4b-3}}{2}=0$$

So $x_1=\frac{-1+\sqrt{4b-3}}{2}$ is the solution to this equation. The rest of the solutions can be shown similarly.

Answer: $x_{1.2}=\frac{-1\pm\sqrt{4b-3}}{2}$, $b\geq\frac{3}{4}$. $x_{3.4}=\frac{1\pm\sqrt{1+4b}}{2}$, $b\geq-\frac{1}{4}$

Example 3. Solve the equation $log_a\left(a+\sqrt{a+x}\right)=\frac{2}{log_x a}$.

Solution. To solve this equation, we first find the range of acceptable values that satisfies its parameter: $x>0, x\neq1, a>0, a\neq1$.

We write it as $log_a\left(a+\sqrt{a+x}\right)=log_a x^2$. According to the exponentiation rule,

$a+\sqrt{a+x}=x^2, \sqrt{a+x}=x^2-a$, where if you square both sides of the equation $x^2>a, a+x=x^4-2ax^2+a^2$, this equation is a quadratic equation in x. We solve it as a quadratic equation with respect to the parameter a: $a^2-(2x^2+1)a+x^4-x=0$, $a_{1.2}=\frac{2x^2+1\pm\sqrt{4x^2+4x+1}}{2}=\frac{2x^2+1\pm(2x+1)}{2}$

Hence $a_1 = x^2 + x + 1$, $a_2 = x^2 - x$. The solution of the equation $a_1 = x^2 + x + 1$ does not lie in the range of acceptable values, since $x^2 - x > 0, x > 0$ we solve the equation $a = x^2 - x$: $x^2 - x - a = 0$, $a \geq -\frac{1}{4}$ hence, $x_{1.2} = \frac{1 \pm \sqrt{1+4a}}{2}, x_1 = \frac{1+\sqrt{1+4a}}{2}, x_2 = \frac{1-\sqrt{1+4a}}{2}$. Of these solutions $x_2 = \frac{1-\sqrt{1+4a}}{2}$ has no roots, since $x < 0$ we show that $x_1 = \frac{1+\sqrt{1+4a}}{2}$ is the root of the given equation:

$$log_a(a + \sqrt{a + x}) = log_a\left(a + \sqrt{a + \frac{1+\sqrt{1+4a}}{2}}\right) = log_a\left(a + \sqrt{\frac{2a+1+\sqrt{4a+1}}{2}}\right) = log_a\left(a + \sqrt{\frac{4a+2+2\sqrt{4a+1}}{4}}\right) = log_a\left(a + \sqrt{(\frac{1+\sqrt{4a+1}}{2})^2}\right) = log_a\left(a + \frac{1+\sqrt{1+4a}}{2}\right) = log_a\left(\sqrt{(\frac{1+\sqrt{4a+1}}{2})^2}\right)$$ satisfies the equation, so it is a solution.

Answer: $x_1 = \frac{1+\sqrt{4a+1}}{2}, a \geq -\frac{1}{4}$

Example 4: Solve the equation $\sqrt{a - \sqrt{a + x}} = x$.

Solution: *Method 1:* From this equation, we can set $\begin{cases} \sqrt{a + x} = t \\ \sqrt{a - t} = x \end{cases} \Rightarrow t \geq 0, x \geq 0$. Squaring the inequalities, we obtain the equalities $\begin{cases} a + x = t^2 \\ a - t = x^2 \end{cases}$. Find *a* from the first equation and substitute it into the second equation. $x + t = t^2 - x^2, x + t - (t^2 - x^2) = 0$,

$$(x + t) + (x^2 - t^2) = 0, (x + t) + (x + t)(x - t) = 0, \begin{cases} x + t = 0 \\ 1 + (x + t) = 0 \end{cases} \Rightarrow \begin{cases} t = -x \\ t = x + 1 \end{cases},$$

$t = -x$, has no solution since $t \geq 0, x \geq 0$. $t = x + 1, \sqrt{a + x} = x + 1$, square the equation. $a + x = x^2 + 2x + 1, x^2 + x + 1 - a = 0, x_{1.2} = \frac{-1 \pm \sqrt{4a-3}}{2}$.

Since $x_1 = \frac{-1-\sqrt{4a-3}}{2}$, $x < 0$, there is no solution. We check that $x_2 = \frac{-1+\sqrt{4a-3}}{2}$ is a solution to the equation. $\sqrt{a - \sqrt{a + \frac{-1+\sqrt{4a-3}}{2}}} = \sqrt{a - \sqrt{\frac{2a-1+\sqrt{4a-3}}{2}}} = \sqrt{a - \sqrt{\frac{4a-2-2\sqrt{4a-3}}{4}}} = \sqrt{a - \sqrt{\frac{(\sqrt{4a-3}+1)^2}{4}}} = \sqrt{a - \frac{\sqrt{4a-3}+1}{2}} = \sqrt{\frac{2a-1-\sqrt{4a-3}}{2}} = \sqrt{\frac{4a-2-2\sqrt{4a-3}}{4}} = \sqrt{\frac{(\sqrt{4a-3}+1)^2}{4}} = \frac{-1+\sqrt{4a-3}}{2}$ satisfies the equation, so

$x_2 = \frac{-1+\sqrt{4a-3}}{2}$ is the solution.

Answer: $= \frac{-1+\sqrt{4a-3}}{2}, \geq \frac{3}{4}$.

Method 2: $\begin{cases} x \geq 0 \\ a + x \geq 0 \\ a - \sqrt{a + x} \geq 0 \end{cases} \Rightarrow \begin{cases} x \geq 0 \\ x \geq -a \\ a \geq \sqrt{a + x} \end{cases} \Rightarrow \begin{cases} x \geq 0 \\ x \geq -a \\ a^2 \geq a + x \end{cases} \Rightarrow \begin{cases} x \geq 0 \\ x \geq -a \\ x \leq a^2 - a \end{cases}$

domain equation is $a^2 - a \geq 0$, hence it follows that $a \geq 1$,

$0 \leq x \leq a^2 - a$. $a - \sqrt{a + x} = x^2$ or $\sqrt{a + x} = a - x^2$ this equation has a solution only if $a - x^2 \geq 0$ so $a + x = a^2 - 2ax^2 + x^4$, is a quadratic equation for *x*. Let's solve it as a quadratic equation for *a*. $a^2 - (2x^2 + 1)a + (x^4 - x) = 0, a_{1.2} = \frac{2x^2+1 \pm \sqrt{(2x^2+1)^2 - 4x^2 + 4x}}{2}$, $a_1 = x^2 - x$, the roots of the equation $x_{1.2} = \frac{1 \pm \sqrt{1+4a}}{2}$ will not be a solution, $a_2 = x^2 + x + 1$, $x_{1.2} = \frac{-1 \pm \sqrt{4a-3}}{2}$ $a \geq \frac{3}{4}, x = \frac{-1-\sqrt{4a-3}}{2}$ cannot be a solution to the given equation, because $x < 0$. This equation satisfies $x = \frac{-1+\sqrt{4a-3}}{2}$, so this is the solution.

Answer: $x = \frac{-1+\sqrt{4a-3}}{2}, a \geq \frac{3}{4}$

Example 5: Solve the equation $x^6 - x^2 - t^2x^2 + t = 0$.

Solution: Since this equation is a sixth power equation in *x*, we solve $x^2t^2 - t + x^2 - x^6 = 0$ as a quadratic equation in *t*. We will find solutions. $t_{1.2} = \frac{1\pm\sqrt{1-4x^2\cdot(x^2-x^6)}}{2} = \frac{1\pm|2x^4-1|}{2}$ is formed. According to the definition of the module $|2x^4-1|$ write as $|2x^4-1| = \begin{cases} 2x^4-1, \left(-\infty; \frac{1}{\sqrt{2}}\right] \cup \left[\frac{1}{\sqrt{2}}; \infty\right) \\ -2x^4+1, -\frac{1}{\sqrt[4]{2}} < x < \frac{1}{\sqrt[4]{2}} \end{cases}$,

we find solutions of the parameter *t* depending on *x*, $\begin{cases} t_1 = \frac{1+2x^4-1}{2x^2} = x^2 \\ t_2 = \frac{1-2x^4+1}{2x^2} = \frac{1-x^4}{x^2} \end{cases}$ x in the equations we express in terms of *t*: $\begin{cases} x^2 = t \\ x^4 + tx^2 - 1 = 0 \end{cases} \Rightarrow \begin{cases} x_{1.2} = \pm\sqrt{t}, t \geq 0 \\ x_{3.4} = \frac{-t\pm\sqrt{1+\sqrt{t^2+4}}}{2} \end{cases}$. $x_{1.2} = \pm\sqrt{t}$ is the solution of this equation $(\sqrt{t})^6 - \left(\sqrt{t}\right)^2 - t^2\left(\sqrt{t}\right)^2 + t = t^3 - t - t^3 + t = 0, 0 = 0$ will be. So the equation will be solved. Similarly, by substituting $x_{3.4}$ into the equation, we can see that this is the solution.

Answer: $x_{1.2} = \pm\sqrt{t}, t \geq 0, x_{3.4} = \frac{-t\pm\sqrt{1+\sqrt{t^2+4}}}{2}$

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