The Educational Effectiveness of Methods for Solving Functional Equations

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Abstract: In this article, we examined one of the methods for solving functional equations, using the most important concept of modern algebra- the concept of a group. And in the article we also showed examples that many functions are obtained from basic ones using compositions and algebraic operations.

Key words: Composition of a function, group, linear system of equations, set of linear functions, composition operation.

Introducsion. You are familiar with functional equations, although you may not know that they are called that.Thus,it is the functions

$$f(x) = f(-x), f(-x) = -f(x)\,, f(x+a) = f(x)$$

that define such properties of functions as evenness,oddnes,and periodicity.

In general, a functional equation is a relation from which you need to find an unknown function.For example,

$$f(x+1) + f(x) = x\,,$$

$$2f(1-x) - xf(x) = -1,$$

$$xf(x) + f\left(\frac{4}{2-x}\right) = x.$$

In this article we will consider one of the methods for solving functional equations,using the most important concept of modern algebra-the concept of group.

Composition of function.

Many functions are obtained from basic ones using compositions and algebraic operations.So,the function $f(x) = \sin(2x+1)$ a composition of the linear function $g(x) = 2x+1$ and trigonometric function $h(x) = sinx$,i,c. $f(x) = h(g(x)) = (h \circ g)(x)$.

The function $f(x) = lgarcsinx$ is obtained as a result of the composition of the functions $g(x) = arcsinx$ and $h(x) = lgx$. Note that the domain of X from D(g) for which $g(x) \in D(h)$. In the last example D(g)=[-1;1] , D(h)=(0;∞).Since $arcsinx$ at $x \in (0; 1]$, that $D(f) = (0; 1]$.

If we take the composition of these same functions in reverse order, that is, the function $f(x) = arcsinlgx$, then we get $D(f) = [\frac{1}{10}\,; 1]$

The composition of the fractional linear functions $g(x) = \frac{-2x+1}{3x+2}$ and

$h(x) = \frac{3x-2}{-x+4}$ is the function $f(x) = h(g(x)) = \frac{3*\frac{-2x+1}{3x+2}-2}{-\frac{-2x+1}{3x+2}+4} = \frac{-12x-1}{14x+7}$,

$x \neq -\frac{2}{3}$ Here $D(f) = R\backslash\{-\frac{2}{3}\ ; -\frac{1}{2}\}$

As a rule,$f \circ g \neq g \circ f$.At the same time,for any functions

$(f \circ g) \circ h = f \circ (g \circ h)$,

which directly follows from the definition of composition.

Let's solve the following problem **Task-1.** Find all functions $y = f(x)$ such that

$2f(-x) - xf(x) = -1$ (1)

Solution. Suppore that there is a function $f(x)$ that satisfies this equation.

Replacing X with 1-X we get

$2f(x) - (1-x)f(1-x) = -1$ (2)

$f(x) = f_1$, $f(1-x) = f_2$ then we get a system of equations

$$\begin{cases} 2f_1 - (1-x)f_2 = -1 \\ -xf_1 + 2f_2 = -1 \end{cases}$$

Solve the system using Cramer's rule

$$\Delta = \begin{vmatrix} 2 & -(1-x) \\ -x & 2 \end{vmatrix} = 4 - x(1-x) = x^2 - x + 4$$

$$\Delta_1 = \begin{vmatrix} -1 & -(1-x) \\ -1 & 2 \end{vmatrix} = -2 - (1-x) = -3 + x$$

$$f_1 = f(x) = \frac{\Delta}{\Delta_1} = \frac{x-3}{x^2 - x + 4}$$

By direct Verification we convinced that the resulting function satisfies equation (1).We reduced the solution of the functional equation to the solution of a system of two linear equations with two unknowns.

Let's now consider a more complex problem.**Task-2** Solve the equations

$xf(x) + 2f\left(\frac{x-1}{x+1}\right) = 1$ (3)

Solution. Let's try to act in the same way as in the first case .Let's replace

$x \rightarrow \frac{x-1}{x+1}$. We get $\frac{x-1}{x+1} f\left(\frac{x-1}{x+1}\right) + 2f\left(-\frac{1}{x}\right) = 1$ (4)

Along with the expressions $f(x)$ and $f\left(\frac{x-1}{x+1}\right)$, we now have a new unknown $f\left(-\frac{1}{x}\right)$. Let's try to apply one more substitution to (3). We have

$-\frac{1}{x} f\left(-\frac{1}{x}\right) + 2f\left(\frac{x+1}{1-x}\right) = 1$ (5)

In addition to $\left(-\frac{1}{x}\right)$, the undesirable expression $f\left(\frac{x+1}{1-x}\right)$ appeared in the equation. Well, let's try substituting $x \rightarrow \frac{x+1}{1-x}$ into (3) and finally, luck.We get the equation.

$\frac{x+1}{1-x} f\left(\frac{x+1}{1-x}\right) + 2f(x) = 1$ (6)

A system of four linear equations (3)-(6) with four unknowns $f(x)$, $f\left(\frac{x-1}{x+1}\right)$, $f\left(-\frac{1}{x}\right)$ and $f\left(\frac{x+1}{1-x}\right)$ has been constructed.Let's put

$$f(x) = x_1\,, f\left(\frac{x-1}{x+1}\right) = x_2\,,\ f\left(-\frac{1}{x}\right) = x_3\,, f\left(\frac{x+1}{1-x}\right) = x_4$$

then we get the following system of equations

$$\begin{cases} x \cdot x_2 + 2x_2 = 1 \\ \dfrac{x-1}{x+1} \cdot x_2 + 2x_3 = 1 \\ -\dfrac{1}{x} \cdot x_3 + 2x_4 = 1 \\ \dfrac{x+1}{x-1} \cdot x_4 + 2x_1 = 1 \end{cases}$$

Solve the system using Cramer's rule

$$\Delta = \begin{vmatrix} x & 2 & 0 & 0 \\ 0 & \frac{x-1}{x+1} & 2 & 0 \\ 0 & 0 & -\frac{1}{x} & 2 \\ 2 & 0 & 0 & \frac{x+1}{1-x} \end{vmatrix} = x \cdot \begin{vmatrix} \frac{x-1}{x+1} & 2 & 0 \\ 0 & -\frac{1}{x} & 2 \\ 0 & 0 & \frac{x+1}{1-x} \end{vmatrix} - 2 \cdot \begin{vmatrix} 0 & 2 & 0 \\ 0 & -\frac{1}{x} & 2 \\ 2 & 0 & \frac{x+1}{1-x} \end{vmatrix} = x \cdot \frac{1}{x} - 2 \cdot 8 = -15$$

$$\Delta_1 = \begin{vmatrix} 1 & 2 & 0 & 0 \\ 1 & \frac{x-1}{x+1} & 2 & 0 \\ 1 & 0 & -\frac{1}{x} & 2 \\ 1 & 0 & 0 & \frac{x+1}{1-x} \end{vmatrix} = \begin{vmatrix} \frac{x-1}{x+1} & 2 & 0 \\ 0 & -\frac{1}{x} & 2 \\ 0 & 0 & \frac{x+1}{1-x} \end{vmatrix} - 2 \cdot \begin{vmatrix} 1 & 2 & 0 \\ 1 & -\frac{1}{x} & 2 \\ 1 & 0 & \frac{x+1}{1-x} \end{vmatrix} = \frac{1}{x} + 2 \cdot \frac{x+1}{x(1-x)} - 8 +$$

$$4 \cdot \frac{x+1}{1-x} = \frac{12x^2-3x+3}{x(1-x)}$$

$$x_1 = \frac{\Delta_1}{\Delta} = \frac{12x^2-3x+3}{-15x(1-x)} = \frac{4x^2-x+1}{5x(x-1)}\ (x \neq -1, x \neq 0\,, x \neq 1.)$$

Verification shows that $f(x)$ satisfies equation (3)

Groups appear.

Let's try to figure out why we were able to solve the equations of the previous paragraph.Let's consider another equation

$$f(x+1) + f(x) = x$$

It looks no more scary that equation (3), but all attempst to solve it in the sam way will in vain: when replacing $x \to x+1$, "the unknown" $f(x+2)$ appears and so on. The chain does not close: we will never get a linear systems.

Recall that when solving the first equation, we performed the substitution $x \to 1-x$ .In this case,$1-x \to x$.That is in relation to the composition operation they behave like this

$g_1 \circ g_2 = g_2 \circ g_1 = g_2\,, g_2 \circ g_2 = g_2\,, g_1 \circ g_1 = g_1$

Consider the "multiplication" table (at the intersection of row number i and column number j there is $g_i \circ g_j$).

$\circ$	g_1	g_2
g_1	g_1	g_2
g_2	g_2	g_1

In each row and each column of this table there are both g_1 and g_2.

Let us now assume that we need to solve the equation

$a(x)f(x) + b(x)f(1-x) = c(x)$ (7)

where a ,b, c are same functions.It is clear that by substituting $x \to 1-x$, we get the equation

$a(1-x)f(1-x) + b(1-x)f(x) = c(1-x)$ (8)

Which together with equation (7) forms a linear system with respect to the functions $f(x)$ and $f(1-x)$.Further, the solution will develop in the same way as when solving

Problem 1.In the second example considered, we made substitutions

$$x \to \frac{x-1}{x+1}, x \to -\frac{1}{x}, x \to \frac{x+1}{1-x}$$

that is , we dealt with the functions

$$g_1(x) = x, g_2(x) = \frac{x-1}{x+1}, g_3(x) = -\frac{1}{x}, g_4(x) = \frac{x+1}{1-x}$$

Let's see how the functions g_1, g_2, g_3, g_4 behave in relation to the composition operation.Let's create table 2 ,similar to table 1 (at the intersection of the i-th row and the k-th column ,write $g_i \circ g_j$)

$\circ$	g_1	g_2	g_3	g_4
g_1	g_1	g_2	g_3	g_4
g_2	g_2	g_3	g_4	g_1
g_3	g_3	g_4	g_1	g_2
g_4	g_4	g_1	g_2	g_3

This table is symmetrical with respect to its diagonal (this means that $g_i \circ g_k = g_k \circ g_i$ for and k and i).

Moreover, all g_i functions appear in every row and every column equally once, and finally,it is easy to notice that

$g_3 = g_2^2, g_4 = g_2^3, g_1 = g_2^4$. Here $g_2^i = g_2 \circ g_2 \circ g_2 \circ g_2 \ldots \circ g_2$

Thus, the system of function $G = \{g_1, g_2, g_3, g_4\}$ has the following properties:

1) It is closed under composition;
2) Among there functions there is an identity mapping $g_1(x) = x$;
3) Each of the functions g_i has an inverse

$g_i^{-1}: g_1^{-1} = g_1, g_2^{-1} = g_4, g_3^{-1} = g_3, g_4^{-1} = g_2$

The system of function $G = \{g_1, g_2\}$ from example 1 has the same properties.

If we were now asked to solve any functional equation of the from

$a(x)f(x) + b(x)f\left(\frac{x-1}{x+1}\right) + c(x)f\left(-\frac{1}{x}\right) + d(x)f\left(\frac{x+1}{1-x}\right) = h(x)$, (9)

we would do this by making the substitutions $x \to g_2(x), x \to g_3(x), x \to g_4(x)$,

after which we would arrive at a linear system.

For example, let's write down what comes of (9) after replacing $x \to g_2(x)$ More over, $g_2(x) \to g_3(x)$, $g_3(x) \to g_4(x), g_4(x) \to g_1(x)$, so we get the equation

$$a\left(\frac{x-1}{x+1}\right)f\left(\frac{x-1}{x+1}\right) + b\left(\frac{x-1}{x+1}\right)f\left(-\frac{1}{x}\right) + c\left(\frac{x-1}{x+1}\right)f\left(\frac{x+1}{1-x}\right) + d\left(\frac{x-1}{x+1}\right)f(x) = h\left(\frac{x-1}{x+1}\right)$$

Now let's give the following definition.

Definition. An arbitrary set G of functions defined on some set M is called a group under the operation $\circ$, if it has the same properties as the system (g_1, g_2, g_3, g_4), that is,

1. For any two functions $f \in G, g \in G$, their composition $f \circ g$ is also belongs to G.
2. The function $e(x) = x$ belongs to G.
3. For every function $f \in G$ belongs is an inverse function f^{-1} , which also belongs to G .

Conclusion.

We can now outline a general method for solving certain functional equations using the concept of a group of functions.Let in the functional equation

$a_1 f(g_1) + a_2 f(g_2) + \ldots + a_n f(g_n) = b$ (10)

The expressions under the sign of the unknown function $f(x)$ be elements of a group G consisting of "n" function: $g_1(x) = x\ , g_2(x)\ , \ldots , g_n(x)$, and the coefficients of equation (10) $a_1, a_2, \ldots, a_n$, b are some functions of x. Let's assume that equation (10) has a solution. Let's replace $x \to g_2(x)$.

As a result, the function sequence $g_1, g_2, \ldots, g_4$ will transform into the sequence $g_1 \circ g_2, g_2 \circ g_2\ , \ldots , g_n \circ g_2$,again consisting of all elements of the group.

Therefore, the "unknowns" $f(g_1), f(g_2), \ldots, f(g_n)$ will be rearranged and we will obtain a new linear equation of the same form as (10).Next, in equation (10) we make the substitutions $x \to g_3(x),\ x \to g_4(x), \ldots., x \to g_n(x)$, after which we obtain a system of n linear equations, that should be solved.If there are solutions,than we must also check to make sure that they satisfy equation (10).

As an example, consider the equation

$2xf(x) + f\left(\frac{1}{1-x}\right) = 2x$ (11)

The set of functions $g_1 = x\ , g_2 = \frac{1}{1-x}\ , g_3 = \frac{x-1}{x}$ forms a group with a multiplication table,

$\circ$	g_1	g_2	g_3
g_1	g_1	g_2	g_3
g_2	g_2	g_3	g_1
g_3	g_3	g_1	g_2

Replacing x in equation (11) by $\frac{1}{1-x}$ and $\frac{x-1}{x}$,we obtain the system

$$\begin{cases} 2xf_1 + f_2 = 2x \\ \dfrac{2}{1-x} f_2 + f_3 = \dfrac{2}{1-x} \\ \dfrac{2(x-1)}{x} f_2 + f_1 = \dfrac{2(x-1)}{x} \end{cases}$$

where $f_1 = f(x), f_2 = f\big(g_2(x)\big) = f\left(\frac{1}{1-x}\right), f_3 = f\big(g_3(x)\big) = f\left(\frac{x-1}{x}\right)$, solving which we get by checking $f_1 = f(x) = \frac{6x-2}{7x}$ at $x \neq -1, x \neq 0$.

In conclusion, we give some examples of groups of functions that can be used in solving functional equations.

$G_1 = \{x, a - x\}\ , G_2 = \left\{x\ , \frac{a}{x}\right\}$ (here and further $a \neq 0$) $G_3 = \left\{x\ , \frac{a}{x}, -x, -\frac{a}{x}\right\}$, $G_4 = \left\{x\ , \frac{1}{x}, -x, -\frac{1}{x}\ , \frac{x-1}{x+1}, \frac{1-x}{x+1}, \frac{x+1}{x-1}, \frac{x+1}{1-x}\right\}$, $G_5 = \left\{x, \frac{a^2}{x}, a - x, \frac{ax}{x-a}, \frac{ax-x^2}{x}\ , \frac{a^2}{a-x}\right\}$, $G_6 = \left\{x, \frac{x\sqrt{3}-1}{x+\sqrt{3}}\ , \frac{x-\sqrt{3}}{x\sqrt{3}+1}\ , -\frac{1}{x}\ , \frac{x+\sqrt{3}}{1-x\sqrt{3}}, \frac{x\sqrt{3}+1}{\sqrt{3}-x}\right\}$.

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